

ZERO-TWO LAW FOR COSINE FAMILIES

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ABSTRACT. For $(C(t))_{t \geq 0}$ being a strongly continuous cosine family on a Banach space, we show that the estimate $\limsup_{t \rightarrow 0^+} \|C(t) - I\| < 2$ implies that $C(t)$ converges to I in the operator norm. This implication has become known as the zero-two law. We further prove that the stronger assumption of $\sup_{t \geq 0} \|C(t) - I\| < 2$ yields that $C(t) = I$ for all $t \geq 0$. Additionally, we give alternative proofs for similar results for C_0 -semigroups.

1. INTRODUCTION

Let $(T(t))_{t \geq 0}$ denote a strongly continuous semigroup on the Banach space X with infinitesimal generator A . It is well-known that the inequality

$$(1.1) \quad \limsup_{t \rightarrow 0^+} \|T(t) - I\| < 1,$$

implies that the generator A is a bounded operator, see e.g. [12, Remark 3.1.4]. Or equivalently, that the semigroup is uniformly continuous (at 0), i.e.,

$$(1.2) \quad \limsup_{t \rightarrow 0^+} \|T(t) - I\| = 0.$$

This has become known as *zero-one* law for semigroups. Surprisingly, the same law holds for general semigroups on semi-normed algebras, i.e., (1.1) implies (1.2), see e.g. [5]. For a nice overview and related results, we refer the reader to [4].

In this paper we study the zero-two law for strongly continuous cosine families on a Banach space, i.e. whether

$$(1.3) \quad \limsup_{t \rightarrow 0^+} \|C(t) - I\| < 2 \quad \text{implies that} \quad \limsup_{t \rightarrow 0^+} \|C(t) - I\| = 0.$$

This implication is known if the Banach space is UMD, see [6, Corollary 4.2], hence, in particular for Hilbert spaces. On the other hand the $0 - 3/2$ law, i.e.

$$\limsup_{t \rightarrow 0^+} \|C(t) - I\| < \frac{3}{2} \quad \text{implies that} \quad \limsup_{t \rightarrow 0^+} \|C(t) - I\| = 0,$$

holds for cosine families on general Banach spaces as was proved by W. Arendt in [1, Theorem 1.1 in Three Line Proofs]. The result even holds without assuming that the cosine family is strongly continuous. In the same work, Arendt poses the question whether the zero-two law holds for cosine families, [1, Question 1.2 in Three Line Proofs]. The following theorem answers this question positively for

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strongly continuous cosine families. For its proof and the definition of a cosine family we refer to Section 2.

Theorem 1.1. *Let $(C(t))_{t \geq 0}$ be a strongly continuous cosine family on the Banach space X . Then*

$$(1.4) \quad \limsup_{t \rightarrow 0^+} \|C(t) - I\| < 2,$$

implies that $\lim_{t \rightarrow 0^+} \|C(t) - I\| = 0$.

By taking $X = \ell^2$ and

$$C(t) = \begin{pmatrix} \cos(t) & 0 & \cdots & \cdots \\ 0 & \cos(2t) & 0 & \cdots \\ \vdots & & \ddots & \end{pmatrix},$$

it is easy to see that this result is optimal. Whether one can get rid of the assumption that the cosine family is strongly continuous remains open.

The zero-one law for semigroups and the zero-two law for cosine families tells something about the behaviour near $t = 0$. Instead of studying the behaviour around zero, we could study the behaviour on the whole time axis. A result dating back to the sixties is the following; for a semigroup the assumption

$$(1.5) \quad \sup_{t \geq 0} \|T(t) - I\| < 1,$$

implies that $T(t) = I$ for all $t \geq 0$, see e.g. Wallen [13] and Hirschfeld [8]. This result seems not to be well-known among researchers working in the area of strongly continuous semigroup. The corresponding result for cosine families, i.e.,

$$(1.6) \quad \sup_{t \in \mathbb{R}} \|C(t) - I\| < 2 \quad \text{implies that} \quad C(t) = I$$

is hardly studied at all. We prove this result for strongly continuous cosine families on Banach spaces. This result is strongly motivated by the recent work of A. Bobrowski and W. Chojnacki. In [3, Theorem 4], they showed that if $r < \frac{1}{2}$, where

$$(1.7) \quad r = \sup_{t \geq 0} \|C(t) - \cos(at)I\|,$$

then $C(t) = \cos(at)I$ for all $t \geq 0$. They used this to conclude that scalar cosine families are isolated points within the space of bounded strongly continuous cosine families acting on a fixed Banach space, equipped with the supremum norm.

Hence we show that for $a = 0$ the r can be chosen be 2, provided C is strongly continuous. We remark that by using the proof idea in [1, Theorem 1.1 in Three Line Proofs] the implication

$$\sup_{t \in \mathbb{R}} \|C(t) - I\| < r \quad \text{implies that} \quad C(t) = I$$

holds for $r < \frac{3}{2}$ for any cosine family. While this paper was being revised, we heard that A. Bobrowski, W. Chojnacki and A. Gregosiewicz showed that for $a \neq 0$ the implication

$$(1.8) \quad \sup_{t \in \mathbb{R}} \|C(t) - \cos(at)I\| < r \quad \text{implies that} \quad C(t) = \cos(at)I$$

holds for general cosine families with $r = \frac{8}{3\sqrt{3}}$. This constant is optimal, as can be directly seen by choosing $C(t) = \cos(3at)I$. In [11] we wrongly claimed that $r = 2$ was the optimal constant.

The lay-out of this paper is as follows. In Section 2 we prove the zero-two law for strongly continuous cosine families, i.e., Theorem 1.1 is proved. In Section 3 we prove the implication (1.6). Furthermore, we give elementary, alternative proofs for strongly continuous semigroups. Throughout the paper, we use standard notation, such as $\sigma(A)$ and $\rho(A)$ for the spectrum and resolvent set of the operator A , respectively. Furthermore, for $\lambda \in \rho(A)$, $R(\lambda, A)$ denotes $(\lambda I - A)^{-1}$.

2. THE ZERO-TWO LAW AT THE ORIGIN

In this section we prove that for the strongly continuous cosine family C on the Banach space X Theorem 1.1 holds; i.e.,

$$\limsup_{t \rightarrow 0^+} \|C(t) - I\| < 2 \quad \text{implies that} \quad \limsup_{t \rightarrow 0^+} \|C(t) - I\| = 0.$$

However, before we do so, we first recall the definition of a strongly continuous cosine family. For more information we refer to [2] or [7].

Definition 2.1. A family $C = (C(t))_{t \in \mathbb{R}}$ of bounded linear operators on X is called a *cosine family* when the following two conditions hold

- (1) $C(0) = I$, and
- (2) For all $t, s \in \mathbb{R}$ there holds

$$(2.1) \quad 2C(t)C(s) = C(t+s) + C(t-s).$$

It is defined to be *strongly continuous*, when for all $x \in X$ and all $t \in \mathbb{R}$ we have

$$\lim_{h \rightarrow 0} C(t+h)x = C(t)x.$$

Similar as for strongly continuous semigroups we can define the infinitesimal generator.

Definition 2.2. Let C be a strongly continuous cosine family, then the *infinitesimal generator* A is defined as

$$Ax = \lim_{t \rightarrow 0} \frac{2(T(t)x - x)}{t^2}$$

with its domain consisting of those $x \in X$ for which this limit exists.

This infinitesimal generator is a closed, densely defined operator. For the proof of Theorem 1.1, the following well-known estimates are needed. For a proof we refer to Lemma 5.5 and 5.6 in [7].

Lemma 2.3. Let C be a strongly continuous cosine family with generator A . Then, there exists $\omega \geq 0$ and $M \geq 1$ such that

$$(2.2) \quad \|C(t)\| \leq Me^{\omega t} \quad \forall t \geq 0.$$

Furthermore, for $\operatorname{Re} \lambda > \omega$ we have $\lambda^2 \in \rho(A)$ and

$$(2.3) \quad \|\lambda^2 R(\lambda^2, A)\| \leq M \cdot \frac{|\lambda|}{\operatorname{Re} \lambda - \omega}.$$

Hence the above lemma shows that the spectrum of A must lie within the parabola $\{s \in \mathbb{C} \mid s = \lambda^2 \text{ with } \operatorname{Re} \lambda = \omega\}$. To study the spectral properties of the points within this parabola, we can use the following lemma.

Lemma 2.4. *Let C be a strongly continuous cosine family on the Banach space X and let A be its generator. Then, for $\lambda \in \mathbb{C}$ and $s \in \mathbb{R}$ there holds*

(1) $S(\lambda, s)$ defined by

$$(2.4) \quad S(\lambda, s)x = \int_0^s \sinh(\lambda(s-t))C(t)x \, dt, \quad x \in X,$$

is a linear and bounded operator on X and its norm satisfies

$$(2.5) \quad \|S(\lambda, s)\| \leq \sup_{t \in [0, |s|]} \|C(t)\| \cdot \frac{\sinh(|s| \operatorname{Re} \lambda)}{\operatorname{Re} \lambda}.$$

(2) *For $x \in X$ we have $S(\lambda, s)x \in D(A)$,*

$$(2.6) \quad (\lambda^2 I - A)S(\lambda, s)x = \lambda(\cosh(\lambda s)I - C(s))x.$$

Furthermore, $S(\lambda, s)A \subset AS(\lambda, s)$.

(3) *The bounded operators $S(\lambda, s)$ and $C(s)x - \cosh(\lambda s)I$ commute.*

(4) *If $\lambda \neq 0$ and $\cosh(\lambda s) \in \rho(C(s))$, then $\lambda^2 \in \rho(A)$ and*

$$(2.7) \quad \begin{aligned} \|R(\lambda^2, A)\| &\leq \frac{1}{|\lambda|} \cdot \|S(\lambda, s)\| \cdot \|R(\cosh(\lambda s), C(s))\| \\ &\leq \sup_{t \in [0, |s|]} \|C(t)\| \cdot \frac{2|s|e^{|s| \operatorname{Re} \lambda}}{|\lambda|} \cdot \|R(\cosh(\lambda s), C(s))\|. \end{aligned}$$

Proof. We begin by showing item 1. Since the cosine family is strongly continuous, the integral in (2.4) is well-defined. Hence $S(\lambda, s)$ is well defined and linear. So it remains to consider

$$\begin{aligned} \|S(\lambda, s)x\| &\leq \sup_{t \in [0, |s|]} \|C(t)\| \cdot \|x\| \cdot \int_0^{|s|} |\sinh(\lambda t)| \, dt \\ &= \sup_{t \in [0, |s|]} \|C(t)\| \cdot \|x\| \cdot \frac{1}{2} \int_0^{|s|} |e^{\lambda t} - e^{-\lambda t}| \, dt \\ &\leq \sup_{t \in [0, |s|]} \|C(t)\| \cdot \|x\| \cdot \frac{e^{|s| \operatorname{Re} \lambda} - e^{-|s| \operatorname{Re} \lambda}}{2 \operatorname{Re} \lambda}. \end{aligned}$$

Since by definition, the last fraction equals $\frac{\sinh(|s| \operatorname{Re} \lambda)}{\operatorname{Re} \lambda}$, the inequality (2.5) is shown.

Item 2. See [10, Lemma4].

Item 3. This is clear, since $C(t)$ and $C(s)$ commute for $s, t \in \mathbb{R}$.

Item 4. We define the bounded operator

$$B = \frac{1}{\lambda} S(\lambda, s) R(\cosh(\lambda s), C(s)).$$

By item 2., we see that $(\lambda^2 I - A)B = I$. By item 3., we get that $B = \frac{1}{\lambda} R(\cosh(\lambda s), C(s))S(\lambda, s)$. Thus, again by 2., $B(\lambda^2 I - A)x = x$ for $x \in D(A)$. Hence, $\lambda^2 \in \rho(A)$ and first inequality of (2.7) follows. By using the power series of the exponential function, it is easy to see that $\frac{\sinh(|s| \operatorname{Re} \lambda)}{\operatorname{Re} \lambda} \leq 2|s|e^{|s| \operatorname{Re} \lambda}$. Combining this with (2.5) gives the second inequality in (2.7). \square

With the use of the above lemma we can show that the spectrum of A is contained in the intersection of a ball and a parabola, provided (1.4) holds, i.e., $\limsup_{t \rightarrow 0^+} \|C(t) - I\| < 2$.

Lemma 2.5. *Let C be a strongly continuous cosine family on the Banach space X with generator A . Assume that there exists $c > 0$ such that*

$$(2.8) \quad \limsup_{t \rightarrow 0^+} \|C(t) - I\| < c < 2.$$

Then, there exists $M_c, r_c > 0$ and $\phi_c \in (0, \frac{\pi}{2})$ such that

$$(2.9) \quad \mathcal{R}_c := \left\{ \lambda^2 \mid \lambda \in \mathbb{C}, |\lambda| > r_c, |\arg(\lambda)| \in \left(\phi_c, \frac{\pi}{2} \right] \right\} \subset \rho(A),$$

and

$$(2.10) \quad \forall \mu \in \mathcal{R}_c \quad \|\mu R(\mu, A)\| \leq M_c.$$

Proof. First, we note that by (2.8) we have that there exists a $t_0 > 0$ such that $\|C(t) - I\| < c$ for all $t \in [0, t_0]$, and by symmetry, for all $t \in (-t_0, 0]$. Since $c < 2$, we conclude that $\frac{1}{2}\|C(t) - I\| < \frac{c}{2} < 1$, hence, $I + \frac{1}{2}(C(t) - I) = \frac{1}{2}(C(t) + I)$ is invertible with $\|(C(t) + I)^{-1}\| < \frac{1}{2-c}$ for all $t \in (-t_0, t_0)$. This implies that $-1 \in \rho(C(t))$. By standard spectral theory it follows that the open ball centered at -1 with radius $\|R(-1, C(t))\|^{-1}$ is included in $\rho(C(t))$. Therefore,

$$(2.11) \quad B_{\frac{2-c}{2}}(-1) \subset B_{\frac{1}{2\|R(-1, C(t))\|}}(-1) \subset \rho(C(t)) \quad \forall t \in (-t_0, t_0),$$

and by the analyticity of the resolvent, we have for $\mu \in B_{\frac{2-c}{2}}(-1)$ and $t \in (-t_0, t_0)$ that

$$(2.12) \quad \begin{aligned} \|R(\mu, C(t))\| &= \left\| \sum_{n=0}^{\infty} (\mu + 1)^n R(-1, C(t))^{n+1} \right\| \\ &\leq 2\|R(-1, C(t))\| < \frac{2}{2-c}. \end{aligned}$$

Since $\cosh(i\pi) = -1$, by continuity there exists ball in the complex plane with center $i\pi$ which is mapped under the cosh inside the ball around -1 . That is, there exists a $\tilde{r} > 0$ such that

$$(2.13) \quad \cosh(B_{\tilde{r}}(i\pi)) \subset B_{\frac{1-c}{2}}(-1).$$

Let $\lambda \in \mathbb{C}$ be such that $|\arg(\lambda)| \leq \frac{\pi}{2}$. We search for $s \in \mathbb{R}$ such that $\lambda s \in B_{\tilde{r}}(i\pi)$. Let $s_\lambda = \frac{\pi \sin(\arg(\lambda))}{|\lambda|}$ be the unique element on the line $\{\lambda s : s \in \mathbb{R}\}$ which is closest to $i\pi$. We have that $|i\pi - \lambda s_\lambda| = \pi \cos(\arg(\lambda))$. Now, choose $\phi_c \in (0, \frac{\pi}{2})$ large enough such that $\pi \cos(\phi_c) < \tilde{r}$ and choose $r_c > 0$ such that $\frac{\pi}{r_c} < t_0$. Then, for all $\lambda^2 \in \mathcal{R}_c$, we have that $\lambda s_\lambda \in B_{\tilde{r}}(i\pi)$ with $s_\lambda \in (-t_0, t_0)$. By (2.13), $\cosh(\lambda s_\lambda) \in B_{\frac{2-c}{2}}(-1)$ and thus,

$$(2.14) \quad \cosh(\lambda s_\lambda) \in \rho(C(s_\lambda)), \quad \text{and} \quad \|R(\cosh(\lambda s_\lambda), C(s_\lambda))\| \leq \frac{2}{2-c},$$

by (2.11) and (2.12). Therefore, 4. of Lemma 2.4 implies that $\lambda^2 \in \rho(A)$ and

$$\begin{aligned} \|R(\lambda^2, A)\| &\leq \sup_{t \in [0, |s_\lambda|]} \|C(t)\| \cdot \frac{2|s_\lambda|e^{|s_\lambda| \operatorname{Re} \lambda}}{|\lambda|} \cdot \|R(\cosh(\lambda s), C(s_\lambda))\| \\ &\leq \sup_{t \in [0, t_0]} \|C(t)\| \cdot \frac{2\pi e^\pi}{|\lambda|^2} \cdot \frac{2}{2-c} \leq \frac{M_c}{|\lambda|^2} \end{aligned}$$

for some M_c only depending on $\sup_{t \in [0, t_0]} \|C(t)\|$ and c . \square

Combining the results from Lemmas 2.3 and 2.5 enable us to prove Theorem 1.1. As for semigroups we can prove a slightly more general result.

Theorem 2.6 (Zero-two law for cosine families). *Let C be a strongly continuous cosine family on the Banach space X . Denote by A its infinitesimal generator. Then the following are equivalent*

(1) *The following inequality holds*

$$\limsup_{t \rightarrow 0^+} \|C(t) - I\| < 2;$$

(2) *The following equality holds*

$$\limsup_{t \rightarrow 0^+} \|C(t) - I\| = 0;$$

(3) *A is a bounded operator.*

Proof. Trivially the second item implies the first one. If the assertion in part 3 holds, then the corresponding cosine family is given by $C(t) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n t^{2n}}{(2n)!}$. From this the property in item 2 is easy to show. Hence it remains to show that item 1 implies item 3.

Let c be the constant from equation (2.8), and let $r_c > 0, \phi_c \in [0, \frac{\pi}{2})$ be the constants from Lemma 2.5. By Lemma 2.3, we have that there exists $\omega' > \omega \geq 0$ such that

$$(2.15) \quad \sup_{\lambda \in R_{\omega'} \cap S_{\phi_c}} \|\lambda^2 R(\lambda^2, A)\| < \infty,$$

where $R_{\omega'} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \omega'\}$ and $S_{\phi_c} = \{\mu \in \mathbb{C} : |\arg \mu| \leq \phi_c\}$. Now, let λ such that $|\lambda| > r_c$ and $|\arg(\lambda)| \in (\phi_c, \frac{\pi}{2}]$. Thus $\lambda^2 \in \mathcal{R}_c$, see (2.9), and so by Lemma 2.5,

$$(2.16) \quad \sup_{\lambda^2 \in \mathcal{R}_c} \|\lambda^2 R(\lambda^2, A)\| < \infty.$$

Let $f(z) = z^2$. It is easy to see that the closure of $\mathbb{C} \setminus (\mathcal{R}_c \cup f(R_{\omega'} \cap S_{\phi_c}))$ is compact. Thus, (2.15) and (2.16) yield that there exists an $R > 0$ such that the spectrum $\sigma(A)$ lies within the open ball $B_R(0)$ and

$$(2.17) \quad \sup_{|\mu| > R} \|\mu R(\mu, A)\| < \infty.$$

Hence we have that $\mu \mapsto R(\mu, A)$ has a removable singularity at ∞ . Since A is closed, this implies that A is a bounded operator, [9, Theorem I.6.13], and therefore part 3 is shown. \square

3. SIMILAR LAWS ON \mathbb{R}

In this previous sections we showed that uniform estimates in a neighbourhood of zero implied additional properties. In this section we study estimates which hold on \mathbb{R} or $(0, \infty)$. We show that by applying a scaling trick, the results can be obtained from the already proved laws. The main theorem of this section is the following.

Theorem 3.1. *The following assertions hold*

(1) *For a semigroup T we have that (1.5) implies that $T(t) = I$ for all $t \geq 0$.*

(2) If the strongly continuous cosine family C on the Banach space X satisfies

$$(3.1) \quad \sup_{t \geq 0} \|C(t) - I\| = r < 2$$

then $C(t) = I$ for all t .

Proof. Since the construction of the proof in the two items is very similar, we concentrate on the second one.

For the Banach space X we define $\ell^2(\mathbb{N}; X)$ as

$$(3.2) \quad \ell^2(\mathbb{N}; X) = \{(x_n)_{n \in \mathbb{N}} \mid x_n \in X, \sum_{n \in \mathbb{N}} \|x_n\|^2 < \infty\}.$$

With the norm

$$\|(x_n)\| = \sqrt{\sum_{n \in \mathbb{N}} \|x_n\|^2}$$

this is a Banach space. On this extended Banach space we define $C_{\text{ext}}(t)$, $t \in \mathbb{R}$ as

$$(3.3) \quad C_{\text{ext}}(t)(x_n) = (C(nt)x_n).$$

Hence it is a diagonal operator with scaled versions of C on the diagonal. To prove that C_{ext} is strongly continuous, we take an arbitrary $x \in \ell^2(\mathbb{N}; X)$ and $t \in \mathbb{R}$. Furthermore, we choose an $\varepsilon > 0$ and a $z = (z_n)$, with only finitely many z_n unequal to zero, such that $\|x - z\| \leq \varepsilon$. By the construction of $\ell^2(\mathbb{N}; X)$ this is always possible. Now we find

$$(3.4) \quad \begin{aligned} & \limsup_{h \rightarrow 0} \|C_{\text{ext}}(t+h)x - C_{\text{ext}}(t)x\| \\ & \leq \limsup_{h \rightarrow 0} [\|C_{\text{ext}}(t+h)x - C_{\text{ext}}(t+h)z\| + \\ & \quad \|C_{\text{ext}}(t+h)z - C_{\text{ext}}(t)z\| + \|C_{\text{ext}}(t)z - C_{\text{ext}}(t)x\|] \\ & \leq 3\|x - z\| + 3\|z - x\| + \\ & \quad \limsup_{h \rightarrow 0} \|C_{\text{ext}}(t+h)z - C_{\text{ext}}(t)z\|, \end{aligned}$$

since by (3.1), the cosine family C_{ext} is bounded by 3. Let N be such that $z_n = 0$ for $n > N$. Then

$$\begin{aligned} & \limsup_{h \rightarrow 0} \|C_{\text{ext}}(t+h)z - C_{\text{ext}}(t)z\|^2 = \\ & \limsup_{h \rightarrow 0} \sum_{n=1}^N \|C(nt+nh)z_n - C(nt)z_n\|^2 = 0, \end{aligned}$$

since C is a strongly continuous cosine family. Combining this with (3.4) we find that

$$\limsup_{h \rightarrow 0} \|C_{\text{ext}}(t+h)x - C_{\text{ext}}(t)x\| \leq 6\varepsilon.$$

Since ε is arbitrarily, we conclude that C_{ext} is a strongly continuous cosine family on $\ell^2(\mathbb{N}; X)$.

Now we estimate the distance from this cosine family to the identity on $\ell^2(\mathbb{N}; X)$ for $t \in (0, 1]$.

$$\begin{aligned} \|C_{\text{ext}}(t) - I\|^2 &= \sup_{\|(x_n)\|=1} \|C_{\text{ext}}(t)(x_n) - (x_n)\|^2 \\ &= \sup_{\|(x_n)\|=1} \sum_{n \in \mathbb{N}} \|C(nt)x_n - x_n\|^2 \\ &\leq \sup_{\|(x_n)\|=1} \sum_{n \in \mathbb{N}} r^2 \|x_n\|^2 = r^2, \end{aligned}$$

where we have used (3.1). In particular, this implies that

$$\limsup_{t \rightarrow 0^+} \|C_{\text{ext}}(t) - I\| < 2.$$

By Theorem 2.6, we conclude that the infinitesimal generator of C_{ext} is bounded. Since $C_{\text{ext}}(t)$ is a diagonal operator, it is easy to see that its infinitesimal generator A_{ext} is diagonal as well. Furthermore, the n 'th diagonal element equals nA . Since n runs to infinity, A_{ext} can only be bounded when $A = 0$. This immediately implies that $C(t) = I$ for all t . \square

From the above proof it is clear that if Theorem 2.6 would hold for non-strongly continuous cosine families, then the strong continuity assumption can be removed from item 2 in the above theorem as well.

As follows from the first item, for semigroups no continuity assumption was needed. As mentioned in the introduction, this can also be proved using operator algebraic result going back to Wallen [13]. In the following subsection, we present some alternative proofs, showing that they can be asked as an exercise in a first course on semigroup theory.

3.1. Elementary proofs for semigroups. We now give some elementary proofs of the following result.

Theorem 3.2. *Let T be a strongly continuous semigroup on the Banach space X , and let A denote its infinitesimal generator. If*

$$(3.5) \quad r := \sup_{t \geq 0} \|T(t) - I\| < 1,$$

then $T(t) = I$ for all $t \geq 0$.

Proof of Theorem 3.2, frequency domain. Since the C_0 -semigroup is bounded by (3.5), $(0, \infty) \subset \rho(A)$ and for $\lambda > 0$ we have that

$$\begin{aligned} \|(\lambda I - A)^{-1}x_0 - \lambda^{-1}x_0\| &= \left\| \int_0^\infty (T(t)x_0 - x_0) e^{-\lambda t} dt \right\| \\ &\leq \int_0^\infty \|T(t)x_0 - x_0\| e^{-\lambda t} dt \leq \frac{r}{\lambda} \|x_0\|, \end{aligned}$$

where we used (3.5). Thus

$$\|\lambda(\lambda I - A)^{-1} - I\| \leq r$$

Since $r < 1$, we know that $I + (\lambda(\lambda I - A)^{-1} - I)$ is boundedly invertible, and the norm of this inverse is less or equal to $(1 - r)^{-1}$. Hence

$$(3.6) \quad \|\lambda^{-1}(\lambda I - A)\| \leq \frac{1}{1 - r}.$$

So for all $\lambda > 0$ we have that $\|I - \lambda^{-1}A\| \leq \frac{1}{1-r}$. This can only hold if $A = 0$. \square

Proof of Theorem 3.2, time domain. In general it holds that

$$(3.7) \quad T(t)x - x = A \int_0^t T(s)x \, ds, \quad t > 0, x \in X.$$

For $t > 0$ let B_t denote the bounded operator $x \mapsto B_t x := \int_0^t T(s)x \, ds$. Since for all $x \in X$,

$$\|x - t^{-1}B_t x\| = \frac{1}{t} \left\| \int_0^t x - T(s)x \, ds \right\| \leq \frac{1}{t} \int_0^t \|x - T(s)x\| \, ds \leq r\|x\|,$$

and $r < 1$, it follows that $t^{-1}B_t$ is boundedly invertible for all $t > 0$ and

$$(3.8) \quad \|tB_t^{-1}\| \leq \frac{1}{1-r} \Leftrightarrow \|B_t^{-1}\| \leq \frac{1}{t(1-r)}.$$

This concludes the proof because by (3.7) and the assumption, $\|AB_t\| \leq 1$,

$$(3.9) \quad \|A\| \leq \|B_t^{-1}\| \stackrel{(3.8)}{\leq} \frac{1}{t(1-r)} \quad \forall t > 0,$$

hence, $A = 0$. \square

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